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# FORCED VIBRATION ANALYSIS OF A PLANAR ELLIPTICAL BEAM

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## ABSTRACT

The scope of this study is to investigate the forced vibration of a planar elliptical Timoshenko beam having rectangular cross-section by using the mixed finite element method. The planar elliptical beam is discretized by a two-noded curvilinear mixed finite element. Each node of the element has 12 DOFs, namely, three translations, three rotations, two shear forces, one axial force, two bending moment and one torque. The finite element matrices are derived by using the exact nodal geometrical parameters (arc length, curvature) of the curvilinear element and these parameters are linearly interpolated through the element. The numerical analysis is performed in the Laplace domain and the results are transformed back to time domain numerically by employing the modified Durbin's transformation algorithm. A planar elliptical beam with fixed at both ends are analyzed. It is subjected to a triangular impulsive type of external dynamic point load acting at the midpoint of beam. Through the analysis, the influence of some parameters (e.g. ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam, the thickness of the cross-section) on the dynamic behavior of the planar elliptical beam is investigated. As far as the author's knowledge, the forced vibration of the planar elliptical beam is an original problem for the literature.

*Keywords: elliptical beam; finite element method; forced vibration; Laplace domain; Timoshenko beam theory*

## 1. INTRODUCTION

Beams are widely used in several fields such as, defense, transportation, aerospace structures *etc*. Curved beams are also preferred in many engineering fields due to architectural or structural reasons, besides straight beams. These needs are required to a wide range of curved element (elliptic, parabola, catenary, cycloid, and circle) having curvature range along the arc length with different section shapes and sizes. Curved beams can either be solved analytically or numerically. Analytical solution procedure has limitations in application. Out of the finite elements, series solution, carryover matrix method, which is the well-known numerical procedures, finite element is the most popular one. In finite elements analysis either displacement or mixed type elements are used. In mixed type elements, besides the displacements and rotations, also forces and moments are the nodal unknowns. Depending on the necessity of the problem mainly two different groups of field equations are used for deriving functional, necessary for the finite element formulations. Huang et al. (1998c) are developed an exact solution for in-plane vibration arches having variable curvature and cross-section. A convergence analysis for the natural frequencies of the semi-elliptical beam is obtained and compared with the literature. Then, the influence of some geometric parameters on the natural frequencies is presented for parabolic arches. Huang et al. (2000) are studied the linear out-of-plane dynamic responses of non-circular plane curves having variable cross-section by extending previous works about uniform curved beams (Huang et al. 1998a) and in-plane dynamic responses (Huang et al. 1998b). Free vibration analysis are investigated in the following non-circular plane curves: parabolic, elliptic, sinusoidal,

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catenary types in Oh et al. (1999) and Oh et al. (2000); parabolic, elliptical, sinusoidal in Yang et al. (2008); elliptical arch in Shahba et al. (2013); parabolic, elliptic, Tschirnhausen′s cubic in Luu et al. (2015); horseshoe elliptic in Lee et al. (2016). The free vibration analysis of composite laminated and sandwich circular and non-circular beams is studied in Ye et al. (2016). Free vibration and stability analysis of elliptic beam is considered in Nieh et.al. (2003). Rajasekaran (2013) solved the static, stability, free and forced vibrations of axially functionally graded tapered circular and non-circular arches by using finite element method. By proposing Generalized Differential Quadrature (GDQ) GDQ-time-stepping algorithm, Tornabene et al. (2016) are investigated for transient dynamic analysis of generally-shaped arches (ellipse, parabola, cycloid, catenary, and circumference) with a varying cross section in undamaged or damaged configuration, under different boundary conditions and external forces.

In this study, the forced vibration of a planar elliptical Timoshenko beam having rectangular cross-section is investigated by using the mixed finite element method. The finite element matrices are derived by using the exact nodal geometrical parameters (arc length, curvature) of the curvilinear element and these parameters are linearly interpolated through the element. The exact geometric functions of arc length and curvature of an elliptical plane curve is derived by using the formulation given in Ermis and Omurtag (2017). The numerical analysis is performed in the Laplace domain and the results are transformed back to time domain numerically by employing the modified Durbin's transformation algorithm (Dubner and Abate 1968, Durbin 1974, Narayanan 1980). The planar elliptical beam with fixed at both ends are analyzed. It is subjected to a triangular impulsive type vertical distributed load. The influence of the ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam and the thickness of the cross-section on the dynamic behavior of the elliptical beam is analyzed. The present procedure of analyze lets to solve much more complex form of dynamic loading, such as time history of an earthquake acting on a high-tech structure.

## 2. FORMULATION

### *2.1 Field Equations and Functional*

The necessary field equations based on Timoshenko beam theory for the isotropic spatial beam (Omurtag and Aköz 1992) are transformed to frequency domain for the dynamic analysis. The field equations in Laplace domain (Eratlı et al. 2014, Ermis et al. 2016) are given in the Frenet coordinate system as follows:

$$\begin{aligned}
-\bar{\mathbf{T}}_{,s} - \bar{\mathbf{q}} + \rho A z^2 \,\bar{\mathbf{u}} &= \mathbf{0} \\
-\bar{\mathbf{M}}_{,s} - \mathbf{t} \times \bar{\mathbf{T}} - \bar{\mathbf{m}} + \rho \mathbf{I}\, z^2 \;\bar{\mathbf{\Omega}} &= \mathbf{0} \\
\bar{\mathbf{u}}_{,s} + \mathbf{t} \times \bar{\mathbf{\Omega}} - \bar{\mathbf{C}}_{\gamma} \bar{\mathbf{T}} &= \mathbf{0} \\
\bar{\mathbf{\Omega}}_{,s} - \bar{\mathbf{C}}_{\kappa} \bar{\mathbf{M}} &= \mathbf{0}
\end{aligned} \tag{1}$$

$z$ is the Laplace transformation parameter, $s$ denotes the axis parameter along the arc length of the spatial beam. $\bar{\mathbf{u}}\,(\bar{u}_t, \bar{u}_n, \bar{u}_b)$, $\bar{\mathbf{\Omega}}\,(\bar{\Omega}_t, \bar{\Omega}_n, \bar{\Omega}_b)$, $\bar{\mathbf{T}}\,(\bar{T}_t, \bar{T}_n, \bar{T}_b)$ and $\bar{\mathbf{M}}\,(\bar{M}_t, \bar{M}_n, \bar{M}_b)$ are the displacement, rotation, force vector and moment vectors in the Laplace space, respectively. $\rho$ is the density of homogeneous material, $A$ is the area of the cross section, $\mathbf{I} = I_t\,\mathbf{t} + I_n\,\mathbf{n} + I_b\,\mathbf{b}$ is the moment of inertia of the cross section, $\bar{\mathbf{q}}$ and $\bar{\mathbf{m}}$ are the distributed external force and moment vectors in the Laplace space, $\bar{\mathbf{C}}_{\gamma} = \mathbf{C}_{\gamma}$ and $\bar{\mathbf{C}}_{\kappa} = \mathbf{C}_{\kappa}$ are the compliance matrices.

$$\mathbf{C}_{\gamma} = \begin{bmatrix} 1/EA & 0 & 0 \\ 0 & 1/GA' & 0 \\ 0 & 0 & 1/GA' \end{bmatrix}, \qquad \mathbf{C}_{\kappa} = \begin{bmatrix} 1/GI_t & 0 & 0 \\ 0 & 1/EI_n & 0 \\ 0 & 0 & 1/EI_b \end{bmatrix} \tag{2}$$

where $A' = A / k'$ and $k'$ is the shear correction factor. $E$ and $G$ are elasticity and shear moduli, respectively. The functional of the structural problem is obtained in the Laplace domain

$$\begin{aligned}\mathbf{I}(\bar{\mathbf{y}}) = & -[\bar{\mathbf{u}}, \bar{\mathbf{T}}_{,s}] + [\mathbf{t} \times \bar{\mathbf{\Omega}}, \bar{\mathbf{T}}] - [\bar{\mathbf{M}}_{,s}, \bar{\mathbf{\Omega}}] - \tfrac{1}{2}[\bar{\mathbf{C}}_{\kappa} \bar{\mathbf{M}}, \bar{\mathbf{M}}] - \tfrac{1}{2}[\bar{\mathbf{C}}_{\gamma} \bar{\mathbf{T}}, \bar{\mathbf{T}}] + \tfrac{1}{2} \rho A z^2 [\bar{\mathbf{u}}, \bar{\mathbf{u}}] + \tfrac{1}{2} \rho z^2 [\mathbf{I} \bar{\mathbf{\Omega}}, \bar{\mathbf{\Omega}}] \\ & - [\bar{\mathbf{q}}, \bar{\mathbf{u}}] - [\bar{\mathbf{m}}, \bar{\mathbf{\Omega}}] + [(\bar{\mathbf{T}} - \hat{\bar{\mathbf{T}}}), \bar{\mathbf{u}}]_{\sigma} + [(\bar{\mathbf{M}} - \hat{\bar{\mathbf{M}}}), \bar{\mathbf{\Omega}}]_{\sigma} + [\hat{\bar{\mathbf{u}}}, \bar{\mathbf{T}}]_{\varepsilon} + [\hat{\bar{\mathbf{\Omega}}}, \bar{\mathbf{M}}]_{\varepsilon}\end{aligned} \tag{3}$$

where square brackets indicate the inner product, the terms with hats are known values on the boundary and the subscripts $\varepsilon$ and $\sigma$ represent the geometric and dynamic boundary conditions, respectively. The detailed formulation of the functional is documented in (Eratlı et al. 2014).

### *2.2 Mixed Finite Element Method*

A two-noded curved element is employed to discretize the beam domain. The curved element has $2 \times 12$ degrees of freedom. The variable vectors per node are $\mathbf{u}, \mathbf{\Omega}, \mathbf{T}, \mathbf{M}$. Linear shape functions are employed for the interpolation. The curvatures are satisfied exactly at the nodal points and linearly interpolated through the element (Eratlı et al. 2014).

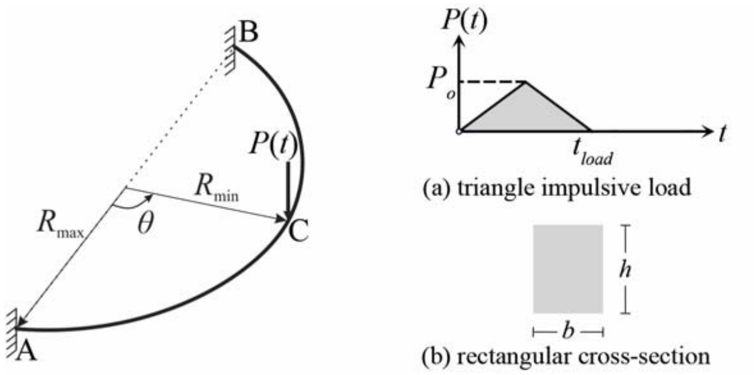



Figure 1. A planar elliptical beam, (a) an impulsive type of external dynamic point load, (b) type of cross-section

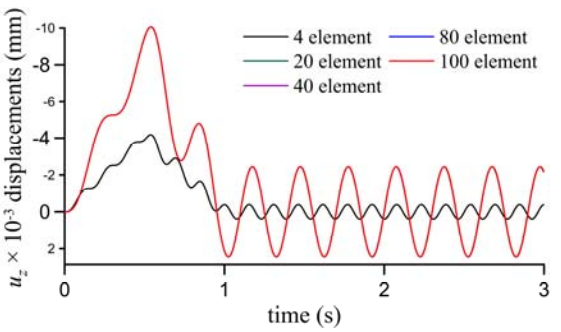
## 3. NUMERICAL EXAMPLES

The forced vibration analysis of a planar elliptical beam is performed to investigate the influence of some parameters (e.g. the ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam, the thickness of the cross-section) on its dynamic behavior. The planar elliptical beam with the rectangular cross-section is subjected to a triangular impulsive type of external dynamic point load $P = P(t)$ acting at the midpoint of beam (Figure 1). The fixed - fixed end condition is employed. Firstly, the convergence test is carried out for the elliptical beam for a ratio $R_{\min} / R_{\max} = 0.9999$ by using the mixed finite element method (MFEM). For verification, the same problem is solved for a planar circular beam ($R_{\min} = R_{\max} = R$) by using MFEM and the commercial program ANSYS. The analyses are carried out in the Laplace space, and the results are transformed back to the time space numerically using modified Durbin's algorithms. The common parameters for the examples are: the modulus of elasticity of the beam is $E = 47.24\,\text{GPa}$, its Poisson's ratio is $\upsilon = 0.2$, the density of material is $\rho = 5000\,\text{kg/m}^3$, the maximum radius of elliptical beam is $R_{\max} = 7.63\,\text{m}$. The width of rectangular cross-section is $b = 0.762\,\text{m}$. The opening angle is $\theta = 180°$. The intensity and the duration of the loading are $P_o = 100\,\text{kN}$ and $t_{load} = 1\text{s}$, respectively. The dynamic response of the beam is determined within $0 \le t \le 3\text{s}$. The parameters used in the analysis for inverse Laplace transformation algorithm are chosen $N = 2^9$ and $aT = 6$ (Eratlı et al. 2014).

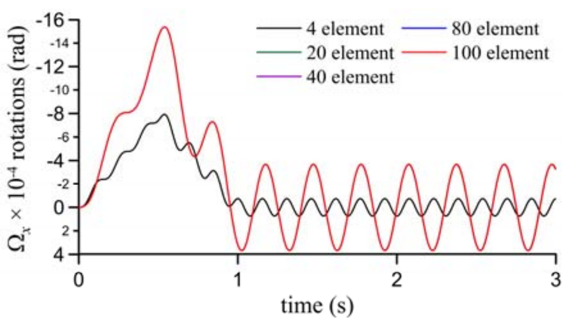
### *3.1 Convergence Test and Verification*

The dynamic analysis of the planar elliptical beam is carried out using 4, 20, 40, 80 and 100 finite elements. As a convergence test and verification example; the height of square cross-section is $h = 0.762\,\text{m}$. The minimum radius of elliptical beam to the maximum radius of elliptical beam ratio is $R_{\min} / R_{\max} = 0.9999$.

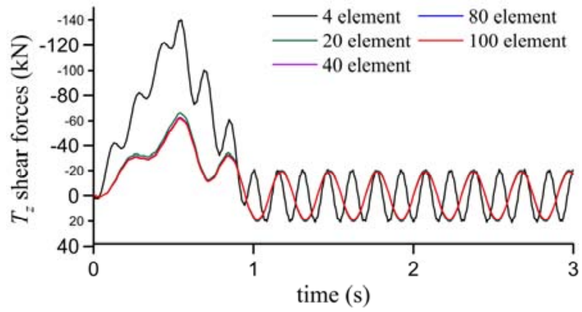
*Convergence test*: The time history curves of the displacement $(u_z)$ and the rotation $(\Omega_x)$ at the midpoint of the beam (at point C) and the shear force $(T_z)$ and the moments $(M_x, M_y)$ at the fixed end of the beam (at point A) are presented in Figure 2. The displacements $(u_z)$ of 80 and 100 elements for $t = 0.5\text{s}$ are compared to the displacement of 40 and 80 elements. The percent differences with respect to the 40 and 80 elements are 0.02% and 0.01, respectively. In the following examples, 80 elements are employed.



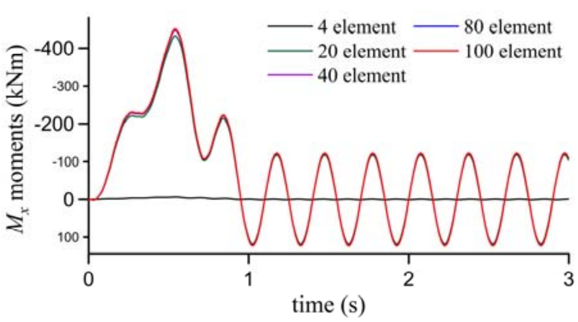

(a) The displacements $u_z$ at the midpoint of the beam



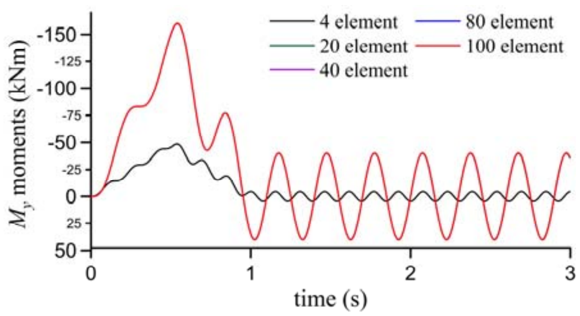

(b) The rotations $\Omega_x$ at the midpoint of the beam




(c) The shear forces $T_z$ at the fixed end of the beam




(d) The moment $M_x$ at the fixed end of the beam




(e) The moments $M_y$ at the fixed end of the beam

Figure 2. Convergence analysis of transverse triangle type impulsive point load applied at the midpoint of the elliptical beam

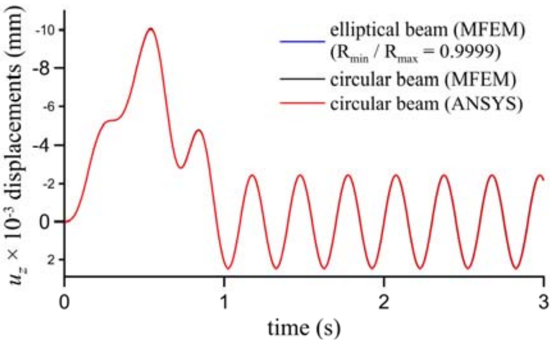
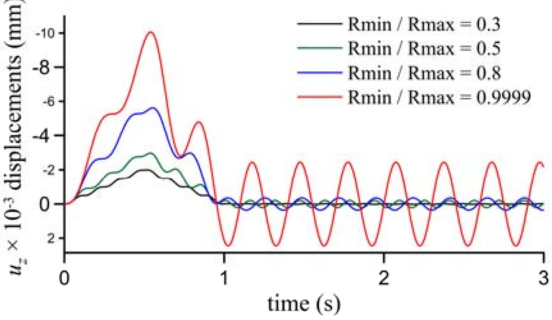



Figure 3. The comparison of the displacements $(u_z)$ at the midpoint of the beam

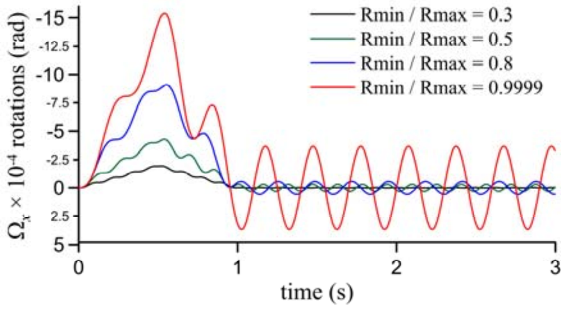
*Verification*: To compare and verify the results obtained for the elliptical beam in Section 3.1, the circular beam having square cross-section which is subjected to a triangular impulsive type of external dynamic point load $P = P(t)$ acting at the midpoint of beam is solved by using MFEM and ANSYS. The results of the elliptical and circular beam are obtained using 80 finite elements. In the ANSYS solution, the circular beam is defined by a total 2032 SOLID-186 elements. The time histories of the displacement $(u_z)$ at the midpoint of the beam (at point C) are given in Figure 3. If the displacements $(u_z)$ of the circular beam MFEM and ANSYS results for $t = 0.5$s is compared with the elliptical beam displacement, the absolute percent differences with respect to the displacement of the elliptical beam are 0.02% and 0.31%, respectively. It is observed that the results are in agreement with each other.

### *3.2 The Dynamic Behavior of Elliptical Beam for Different $R_{min}/R_{max}$ Ratios*

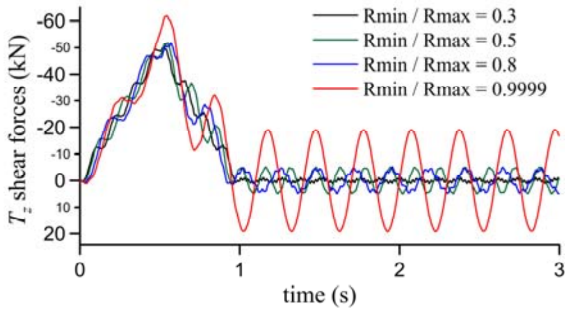
The influence of the ratio $(R_{\min} / R_{\max})$ of the minimum radius of elliptical beam to the maximum radius of elliptical beam on the dynamic behavior of the elliptical beam is analyzed. Keeping the value of the maximum radius as $R_{\max} = 7.63\,\text{m}$, the ratios $(R_{\min} / R_{\max})$ are chosen $0.3, 0.5, 0.8, 0.9999$, respectively. The dimensions of rectangular cross-section are $h = b = 0.762\,\text{m}$. For four different $R_{\min} / R_{\max}$ ratios, the time histories of the displacement $(u_z)$ and the rotation $(\Omega_x)$ at the midpoint of the beam (at point C) and the shear force $(T_z)$ and the moments $(M_x, M_y)$ at the fixed end of the beam (at point A) are presented in Figure 4. As the ratio of the minimum radius of elliptical beam-to-the maximum radius of elliptical beam radius increases, an increasing trend except $M_y$ is observed for $u_z, \Omega_x, T_z, M_x$. The results ($u_z, \Omega_x, T_z, M_x$) of $R_{\min} / R_{\max} = 0.3$ for $t = 0.5$s are compared with the results ($u_z, \Omega_x, T_z, M_x$) of $R_{\min} / R_{\max} = 0.5, 0.8, 0.9999$, and the percent increases in the displacement $(u_z)$, the rotation $(\Omega_x)$, the shear force $(T_z)$ and the moment $(M_x)$ are tabulated in Table 1.



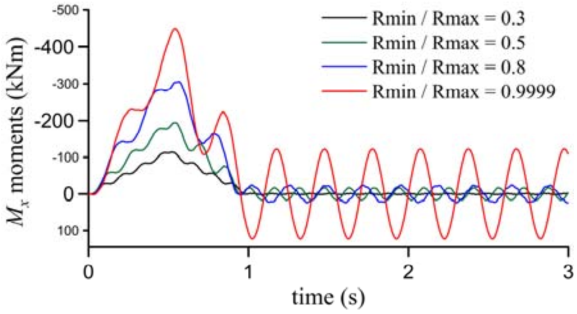

(a) The displacements $u_z$ at the midpoint of the beam



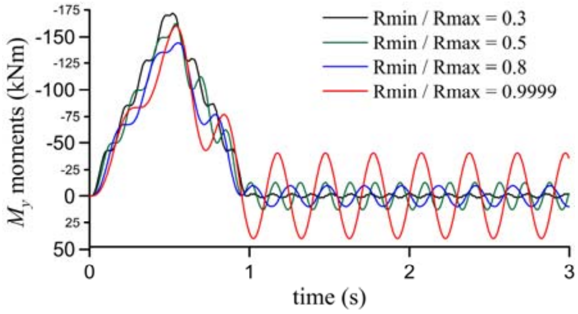

(b) The rotations $\Omega_x$ at the midpoint of the beam




(c) The shear forces $T_z$ at the fixed end of the beam




(d) The moment $M_x$ at the fixed end of the beam




(e) The moments $M_y$ at the fixed end of the beam

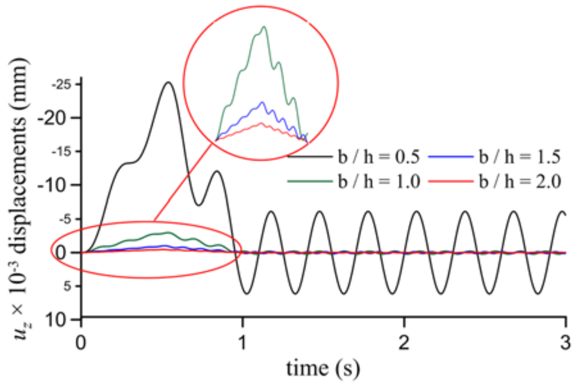
Figure 4. Time histories of the elliptical beam for different values of $R_{\min} / R_{\max}$

Table 1. The percent increases in the displacements, the rotations, the forces and the moments for $t = 0.5\text{s}$ in the case of $R_{\min} / R_{\max} = 0.5, 0.8, 0.9999$ with respect to $R_{\min} / R_{\max} = 0.3$.

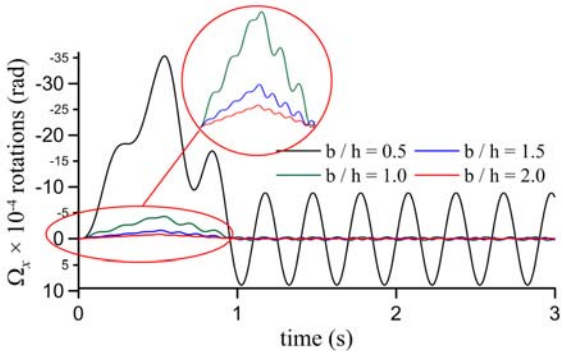
| $R_{\min} / R_{\max}$ | $u_z$ (%) | $\Omega_x$ (%) | $T_z$ (%) | $M_x$ (%) |
|---|---|---|---|---|
| 0.5 | -42.5 | -110.4 | 3.0 | -62.0 |
| 0.8 | -169.9 | -348.4 | 4.6 | -154.8 |
| 0.9999 | -380.9 | -658.6 | -14.9 | -274.6 |

### *3.3 The Dynamic Behavior of Thick Elliptical Beam*

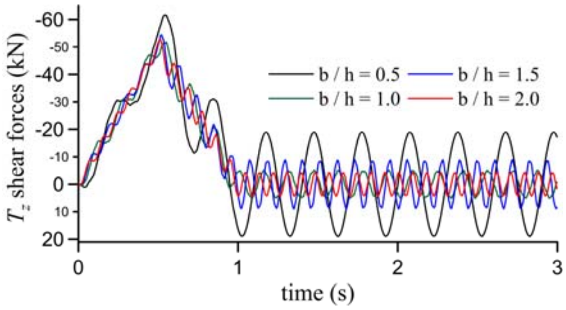
The influence of the thickness of rectangular cross-section on the dynamic behavior of the elliptical beam is analyzed. Keeping the width of the cross-section as $b = 0.762\,\text{m}$, the ratios of $b / h$ are $0.5, 1.0, 1.5, 2.0$, respectively. The ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam is $R_{\min} / R_{\max} = 0.5$. For four different $b / h$ ratios, the time histories of the displacement $(u_z)$ and the rotation $(\Omega_x)$ at the midpoint of the beam (at point C) and the shear force $(T_z)$ and the moments $(M_x, M_y)$ at the fixed end of the beam (at point A) are presented in Figure 5. As the thickness of cross-section increases, an decreasing trend is observed for $u_z, \Omega_x$. The results $(u_z, \Omega_x)$ of $b / h = 0.5$ for $t = 0.5\text{s}$ are compared with the results $(u_z, \Omega_x)$ of $b / h = 1.0, 1.5, 2.0$, and the percent reduction in the displacement $(u_z)$ and the rotation $(\Omega_x)$ are listed in Table 2.



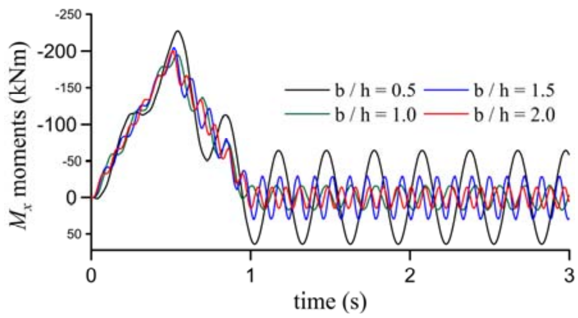

(a) The displacements $u_z$ at the midpoint of the beam



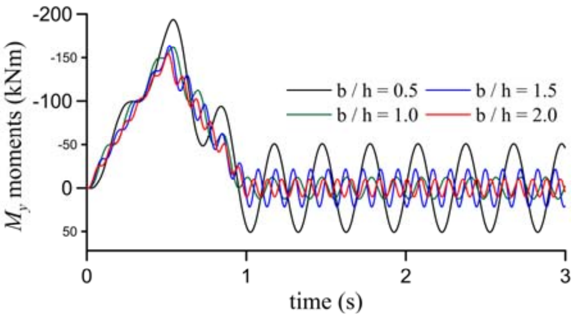

(b) The rotations $\Omega_x$ at the midpoint of the beam




(c) The shear forces $T_z$ at the fixed end of the beam




(d) The moment $M_x$ at the fixed end of the beam




(e) The moments $M_y$ at the fixed end of the beam

Figure 5. Time histories of the elliptical beam for different values of $b/h$

Table 2. The percent increases in the displacements and the rotations for $t = 0.5\text{s}$ in the case of $b/h = 1.0, 1.5, 2.0$ with respect to $b/h = 0.5$.

| $b/h$ | $u_z$ (%) | $\Omega_x$ (%) |
|---|---|---|
| 1.0 | 88.2 | 87.8 |
| 1.5 | 95.9 | 95.3 |
| 2.0 | 98.0 | 97.5 |

## 4. CONCLUSIONS

Dynamic behavior of a planar elliptical Timoshenko beam having rectangular cross-section is investigated using the mixed finite element method. The exact formulation of arc length and curvature of an elliptical plane curve is derived by using the formulation given in Ermis and Omurtag (2017). The solutions are obtained in Laplace space and the results are transformed back to time space by using modified Durbin's algorithm. A convergence test is carried out for the planar elliptical beam which is almost the planar circular beam and its results are compared with the mixed finite element results of a planar circular beam and the results of the commercial program ANSYS. Then, numerical solutions are performed to analyze the dynamic behavior of the planar elliptical beam. Parametric studies are carried out to investigate the influence of the ratio of the minimum radius of elliptical beam to the maximum radius of elliptical beam $(R_{\min} / R_{\max})$, the ratio of the width of rectangular cross-section to the height of rectangular cross-section $(b/h)$. Following remarks can be cited:

- The MFEM results of the planar elliptical beam are verified through the use of two sample problem (the MFEM results of the planar circular beam and the results of ANSYS).
- As the ratio $(R_{\min} / R_{\max})$ increases, an increase in the displacements $(u_z)$, the rotations $(\Omega_x)$, the shear forces $(T_z)$ and the moments $(M_x)$ is observed.
- An increase in the thickness of the rectangular cross-section caused a reduction of the displacement $(u_z)$ and the rotation $(\Omega_x)$.

## 5. ACKNOWLEDGMENTS

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